*J. Fluid Mech.* (2002), *vol.* 450, *pp.* 201–205. © 2002 Cambridge University Press 201
DOI: 10.1017/S0022112001006498 Printed in the United Kingdom

# Spatial versions of the Zakharov and Dysthe evolution equations for deep-water gravity waves

By ELIEZER KIT AND LEV SHEMER

Department of Fluid Mechanics and Heat Transfer, Faculty of Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

(Received 5 August 2001 and in revised form 11 September 2001)

A spatial two-dimensional version of the Zakharov equation describing the evolution of deep-water gravity waves is used to derive two fourth-order evolution equations, for the amplitudes of the surface elevation and of the velocity potential. The scaled form of the equations is presented.

## 1. Introduction

Zakharov (1968) derived an equation which describes the temporal evolution of deep-water waves in the wave-vector Fourier space. This equation is valid to the third order in the wave steepness $\epsilon$ and does not have any restrictions on the spectral width. It was used in the same paper to derive the cubic Schrödinger equation which describes the temporal evolution of the wave envelope under the assumption of a narrow spectrum. The cubic Schrödinger equation was subsequently derived by a multiple-scale perturbation method and applied to studies of water waves by numerous investigators; for references see e.g. Mei (1989). Unsatisfied by the unfavourable comparison of the solution of the cubic Schrödinger equation with the exact computations, Dysthe (1979) extended the perturbation analysis to the fourth-order in $\epsilon$. Stiassnie (1984) obtained the fourth-order modified nonlinear Schrödinger (MNLS) equation from the third order (in the wave amplitude) Zakharov equation assuming the narrow spectrum approximation. This became possible since the fourth order in the Dysthe equation emerges only due to the narrow spectrum approximation, as mentioned by Stiassnie. Extension of the Dysthe MNLS equation to arbitrary water depth was carried out by Brinch-Nielsen & Jonsson (1986) by applying the perturbation expansion method.

Hogan (1985, 1986) followed the approach of Stiassnie to derive from the Zakharov equation two separate fourth-order evolution equations in the physical space for gravity–capillary waves. He distinguished between the evolution equations for the amplitude of the velocity potential, as obtained by Dysthe (1979), and for the surface elevation. Lo & Mei (1985) introduced a transformation of the Dysthe equation which allows the study of the spatial evolution of unidirectional nonlinear wave groups in scaled variables. Their numerical solutions were in qualitative agreement with experimental observations, exhibiting strong front–tail asymmetry of the group envelope. The broader band (BMNLS) equation was obtained by Trulsen & Dysthe (1996, 1997) by expanding the linear part of the equation to higher order in the spectral width. A further attempt in this direction was undertaken by Trulsen *et al.* (2000). They indicated that their model could be extracted from the Zakharov (1968) equation.

A quantitative comparison with the experiments carried out in laboratory tanks, as well as modelling of wave propagation in the coastal zone, requires application of equations describing the evolution in space. A unidirectional spatial version of the Zakharov equation was derived in Shemer *et al.* (2001). Numerical solutions of this equation compared favourably with the experimental results. Qualitative and quantitative agreement was obtained between the numerical simulations and the measurements of the variation of the wave group shapes and the amplitude spectra of the surface elevation along the tank. It is quite natural to use the spatial form of the Zakharov equation to derive the fourth- and higher-order amplitude evolution equations in physical space. This is carried out in § 2.

## 2. Model derivation

The Zakharov equation which describes the slow temporal evolution of the complex amplitude $B(\boldsymbol{k},t)$ of a nonlinear wave field in the wave-vector Fourier space can be presented in a polar coordinate system as

$$\mathrm{i}\frac{\partial B}{\partial t} = \int_0^{\infty} \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \int_{-\pi}^{\pi} k_1\,\mathrm{d}\theta_1 k_2\,\mathrm{d}\theta_2 k_3\,\mathrm{d}\theta_3\, T(k,\theta,k_1,\theta_1,k_2,\theta_2,k_3,\theta_3)$$
$$\times B^*(k_1,\theta_1)B(k_2,\theta_2)B(k_3,\theta_3)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{e}^{-\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t}. \quad (2.1)$$

This presentation allows the generalization of the unidirectional spatial version of the Zakharov equation, Shemer *et al.* (2001), to the horizontal plane $\boldsymbol{x} = (x, y)$:

$$i(\boldsymbol{c}_g\boldsymbol{\cdot}\nabla_h)B(\omega,\theta,\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3 \int_{-\pi}^{\pi} \mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3\, T(\omega,\theta,\omega_1,\theta_1,\omega_2,\theta_2,\omega_3,\theta_3)$$

$$k_1k_2k_3B^*(\omega_1,\theta_1)B(\omega_2,\theta_2)B(\omega_3,\theta_3)\frac{1}{2\pi k}\delta(\omega+\omega_1-\omega_2-\omega_3)\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{x}}, \quad (2.2)$$

where the group velocity vector for deep water $\boldsymbol{c}_g = [\omega 2k(\omega)]\boldsymbol{k}/k$ and the horizontal operator $\nabla_h = (\partial/\partial x/\partial/\partial y)$, $\boldsymbol{k}$ being the wave vector. Equation (2.2) describes the slow spatial evolution of nonlinear waves satisfying conditions of near resonance: $\omega+\omega_1-\omega_2-\omega_3 = 0$; $|\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3| = O(\epsilon^2)$, where $\epsilon$ is the nonlinearity parameter representing the wave steepness. The interaction coefficient $T$ is given in a relatively compact form for precise resonance conditions by Zakharov (1999). For the near resonating quartets, the exact value of the interaction coefficient deviates from this value by a term of order $\epsilon^2$ (Krasitskii & Kalmykov 1993; Annenkov, private communication). Hence, equation (2.2) with the expression for $T$ given in Zakharov (1999) accurately describes the evolution of a wave field with a spectrum of arbitrary width at the third order in wave steepness. Equation (2.2) is now used to derive the fourth-order spatial version of the Dysthe equation, invoking the narrow spectrum approximation.

The free surface elevation $\eta(\boldsymbol{x},t)$ and the velocity potential at the free surface $\psi(\boldsymbol{x},t) = \phi(\boldsymbol{x},\eta,t)$ are related at the leading order to the complex amplitudes $B(\omega,\theta)$:

$$\eta(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\int_{-\pi}^{\pi}k(\omega)\mathrm{d}\theta\left(\frac{\omega}{2g}\right)^{1/2}\{B(\omega,\theta)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]+\mathrm{c.c}\}, \quad (2.3a)$$

$$\psi(\boldsymbol{x},t) = -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\int_{-\pi}^{\pi}k(\omega)\mathrm{d}\theta\left(\frac{g}{2\omega}\right)^{1/2}\{B(\omega,\theta)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]-\mathrm{c.c}\}. \quad (2.3b)$$

The narrow spectrum case is considered, $\omega = \omega_0 + \Omega$, and $\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{\chi}$, where

$\Omega/\omega_0 = O(\epsilon), \boldsymbol{k}_0 = (k_0, 0), |\,\boldsymbol{\chi}|/k_0 = O(\epsilon)$ and $\boldsymbol{\chi} = (\chi, k_0\vartheta), \vartheta = O(\epsilon)$. A new variable is now introduced:

$$A(\Omega, \vartheta, \boldsymbol{x}) = B(\omega, \theta, \boldsymbol{x}) \exp[\mathrm{i}(k_x(\omega_0 + \Omega, \vartheta) - k_0)x]. \tag{2.4}$$

Substitution of (2.4) into (2.2) and (2.3) yields

$$\begin{aligned}
&\mathrm{i}\frac{\partial A(\Omega, \vartheta, \boldsymbol{x})}{\partial x} + [k_x(\omega_0 + \Omega, \vartheta) - k_0]A(\Omega, \vartheta, \boldsymbol{x}) \\
&\quad = \frac{2k_x(\omega_0 + \Omega, \vartheta)}{\omega_0 + \Omega}\frac{k_0^2}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\vartheta_1 \mathrm{d}\vartheta_2 \mathrm{d}\vartheta_3 \int_{-\infty}^{\infty} \mathrm{d}\Omega_2 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3 \\
&\quad \times T(\omega_0 + \Omega, \vartheta, \omega_0 + \Omega_1, \vartheta_1, \omega_0 + \Omega_2, \vartheta_2, \omega_0 + \Omega_3, \vartheta_3)\mathrm{e}^{-\mathrm{i}k_0(\vartheta + \vartheta_1 - \vartheta_2 - \vartheta_3)y} \\
&\quad \times A^*(\Omega_1, \vartheta_1)A(\Omega_2, \vartheta_2)A(\Omega_3, \vartheta_3)\delta(\Omega + \Omega_1 - \Omega_2 - \Omega_3).
\end{aligned} \tag{2.5}$$

The spatially evolving complex amplitudes $a_\eta$ and $a_\psi$ are related to $\eta(\boldsymbol{x}, t)$ and $\psi(\boldsymbol{x}, t)$:

$$\eta(\boldsymbol{x}, t) = \mathrm{Re}[a_\eta(\boldsymbol{x}, t) \exp \mathrm{i}(k_0 x - \omega_0 t)], \tag{2.6a}$$

$$\psi(\boldsymbol{x}, t) = \mathrm{Re}[-\mathrm{i}\frac{g}{2\omega_0}a_\psi(\boldsymbol{x}, t) \exp \mathrm{i}(k_0 x - \omega_0 t)]. \tag{2.6b}$$

Following the approach suggested by Stiassnie (1984), the relation between the amplitudes $a_\eta$ and $a_\psi$ to the variable $A(\Omega, \vartheta, \boldsymbol{x})$ is obtained:

$$a_\eta(\boldsymbol{x}, t) = \frac{k_0}{2\pi}\left(\frac{2\omega_0}{g}\right)^{1/2} \int_{-\pi}^{\pi} \mathrm{d}\vartheta \int_{-\infty}^{\infty} \left[\left(1 + \frac{\Omega}{2\omega_0}\right) A(\Omega, \vartheta, \boldsymbol{x})\mathrm{e}^{-\mathrm{i}\Omega t}\mathrm{e}^{\mathrm{i}k_0\vartheta y}\mathrm{d}\Omega\right], \tag{2.7a}$$

$$a_\psi(\boldsymbol{x}, t) = \frac{k_0}{2\pi}\left(\frac{2\omega_0}{g}\right)^{1/2} \int_{-\pi}^{\pi} \mathrm{d}\vartheta \int_{-\infty}^{\infty} \left[\left(1 - \frac{\Omega}{2\omega_0}\right) A(\Omega, \vartheta, \boldsymbol{x})\mathrm{e}^{-\mathrm{i}\Omega t}\mathrm{e}^{\mathrm{i}k_0\vartheta y}\mathrm{d}\Omega\right]. \tag{2.7b}$$

The wavenumber difference on the left-hand side of (2.5) can be expanded as

$$k_x(\omega_0 + \Omega, \vartheta) - k_0 = \frac{2k_0}{\omega_0}\Omega + \frac{k_0\Omega^2}{\omega_0^2} - \frac{k_0}{2}\vartheta^2 + \frac{k_0}{\omega_0}\vartheta^2\Omega + O(\epsilon^4). \tag{2.8}$$

Note that for a unidirectional case ($\vartheta \equiv 0$), the expansion (2.8) to the second order in $\Omega$ is exact due to the deep-water dispersion relation ($k = \omega^2/g$), in contrast to an infinite number of terms required for the expansion of the frequency difference in Stiassnie (1984) and Hogan (1985), where $\omega = (gk)^{1/2}$.

The small parameter of the problem $\epsilon = a_0k_0$, where $a_0$ is the maximum wave amplitude. Dimensionless scaled variables are now introduced:

$$\left.\begin{aligned}
&a_\eta = a_0A_\eta, \quad a_\psi = a_0A_\psi, \quad \phi = \omega_0a_0^2\Phi, \\
&\epsilon\omega_0\left(\frac{2k_0}{\omega_0}x - t\right) = \tau, \quad \epsilon k_0 y = Y, \quad \epsilon^2k_0x = X, \quad \epsilon k_0 x = Z.
\end{aligned}\right\} \tag{2.9}$$

After expansion of the coefficient $T$ to the first-order in $\Omega$ (there are no first-order terms in the expansion of $T$ in $\vartheta$), and following again the procedure outlined in Stiassnie (1984), two dimensionless evolution equations in the physical space are obtained:

$$\begin{aligned}
&\frac{\partial A_\eta}{\partial X} + \mathrm{i}\frac{\partial^2 A_\eta}{\partial \tau^2} - \frac{\mathrm{i}}{2}\frac{\partial^2 A_\eta}{\partial Y^2} + \epsilon\frac{\partial^3 A_\eta}{\partial\tau\partial Y^2} + \mathrm{i}|A_\eta|^2A_\eta + 8\epsilon|A_\eta|^2\frac{\partial A_\eta}{\partial\tau} \\
&\qquad + 2\epsilon A_\eta^2\frac{\partial A_\eta^*}{\partial\tau} + 4\mathrm{i}\epsilon A_\eta\frac{\partial\Phi}{\partial\tau}\bigg|_{Z=0} = 0,
\end{aligned} \tag{2.10a}$$

$$\frac{\partial A_\psi}{\partial X} + \mathrm{i}\frac{\partial^2 A_\psi}{\partial \tau^2} - \frac{\mathrm{i}}{2}\frac{\partial^2 A_\psi}{\partial Y^2} + \epsilon\frac{\partial^3 A_\psi}{\partial \tau \partial Y^2} + \mathrm{i}|A_\psi|^2 A_\psi + 8\epsilon|A_\psi|^2\frac{\partial A_\psi}{\partial \tau} + 4\mathrm{i}\epsilon A_\psi \left.\frac{\partial \Phi}{\partial \tau}\right|_{Z=0} = 0. \qquad (2.10b)$$

The scaled velocity potential $\Phi$ satisfies

$$4\frac{\partial^2 \Phi}{\partial \tau^2} + \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} = 0, \quad -\epsilon k_0 h < Z < 0 \qquad (2.11)$$

with the boundary conditions

$$\left.\frac{\partial \Phi}{\partial Z}\right|_{Z=0} = \frac{\partial |A_\psi|^2}{\partial \tau} \quad \left.\frac{\partial \Phi}{\partial Z}\right|_{Z=-\epsilon k_0 h} = 0. \qquad (2.12)$$

At this order, $A_\eta$ can be used in (2.12) as well. For the spectral width of the order of $\epsilon$ adopted in our study, equations (2.10*a*, *b*) are equivalent to (2.16) and (2.10) in Trulsen & Dysthe 1997 up to the fourth order. (In Trulsen & Dysthe 1997, the last four terms are of higher order if the present assumption on the spectral width is adopted.)

## 3. Conclusions

The Dysthe equations for the amplitudes of both surface elevation and the velocity potential were obtained from the spatial version of the two-dimensional Zakharov equation. Since the interaction coefficient $T$ is known for the intermediate water depth as well, the present derivation can be easily extended to arbitrary water depth. For the two-dimensional case, the series expansion in the linear term in (2.5) can be easily carried out to any desired order, thus allowing the adoption of the approach of Trulsen *et al.* (2000) to the spatial evolution case. The increased accuracy of the expansion, however, does not generate additional terms in the unidirectional evolution equation, since for the one-dimensional case, there are no terms beyond quadratic. This explains the observation of Trulsen & Dysthe (1997) that in this case the broad-band NLS equation is equivalent to the corresponding Dysthe equation. An expansion of the nonlinear part of the equations to the order beyond $O(\epsilon^4)$ in the spectral width is possible using both temporal and spatial versions of the Zakharov equation. However, care then should be taken to account for the additional terms in the interaction coefficient $T$ that arise when the exact resonance conditions are not satisfied, Krasitskii & Kalmykov (1993). For weakly nonlinear Rossby waves, a higher-order NLS was obtained by Luo (2001) using the perturbation expansion method. Generalization of his higher-order model to water gravity waves requires accounting for terms related to the variation of the induced mean current.

The ocean wave spectrum is usually quite narrow. Equations (2.10) to (2.12) can therefore be used to describe the spatial evolution of propagating waves. When the initial conditions are provided in terms of surface elevation (e.g. are measured by directional a buoy or by an array of pressure transducers), the use of the equation for surface elevation is appropriate. For cases when velocities are measured directly, by either current meters or some remote sensing technique, like an along-track interferometric SAR, the use of the potential amplitude equation becomes more straightforward.

The authors are grateful to M. Stiassnie and S. Annenkov for their suggestions. This research has been supported in part by INTAS.

## REFERENCES

Brinch-Nielsen, U. & Jonsson, I. G. 1986 Fourth order evolution equations and stability analysis for Stokes waves on arbitrary water depth. *Wave Motion* **8**, 455–472.

Dysthe, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

Hogan, S. J. 1985 The fourth-order evolution equation for deep-water gravity-capillary waves. *Proc. R. Soc. Lond.* A **402**, 359–372.

Hogan, S. J. 1986 The potential form of the fourth-order evolution equation for deep-water gravity-capillary waves. *Phys. Fluids* **29**, 3479–3480.

Krasitskii, V. P. & Kalmykov, V. A. 1993 Four-wave reduced equations for surface gravity waves. *Izv. Atm. Ocean. Phys.* **29**, 222–228.

Lo, E. & Mei, C. C. 1985 A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation. *J. Fluid Mech.* **150**, 395–416.

Luo, D. 2001 Derivation of a higher order nonlinear Schrödinger equation for weakly nonlinear Rossby waves. *Wave Motion* **33**, 339–347.

Mei, C. C. 1989 *The Applied Dynamics of Ocean Surface Waves*. World Scientific.

Shemer, L., Jiao, H.-Y., Kit, E. & Agnon, Y. 2001 Evolution of a nonlinear wave field along a tank: experiments and numerical simulations based on the spatial Zakharov equation. *J. Fluid Mech.* **427**, 107–129.

Stiassnie, M. 1984 Note on the modified nonlinear Schrödinger equation for deep water waves *Wave Motion* **6**, 431–433.

Trulsen, K. & Dysthe, K. B. 1996 A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water. *Wave Motion* **24**, 281–289.

Trulsen, K. & Dysthe, K. B. 1997 Frequency downshift in three-dimensional wave trains in a deep basin. *J. Fluid Mech.* **352**, 359–373.

Trulsen K., Kliakhandler, I., Dysthe, K. B & Velarde, M. G. 2000 On weakly nonlinear modulation of waves on deep water. *Phys. Fluids* **12**, 2432–2437.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 190–194.

Zakharov, V. 1999 Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid. *Eur. J. Mech.* B/*Fluids* **18**, 327–344.